Math 261B Tues 9/22

$GL_n \qquad SL_n = \{ g \in GL_n \mid \det g = 1 \}$

$\qquad\qquad PGL_n = GL_n / k^\times I \qquad (= PSL_n)$

$\mathfrak{gl}_n \qquad \mathfrak{sl}_n = \{ x \in \mathfrak{gl}_n \mid \operatorname{tr} X = 0 \} \qquad \mathfrak{pgl}_n = \mathfrak{gl}_n / k \cdot I$

all have 'same' root space decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad R = \{ \underbrace{e_i - e_j}_{= \alpha} \}$$

and root SL_2's

$$SL_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & a & & b & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & c & & d & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \\ \quad\;\; \uparrow \quad \uparrow \\ \quad\;\; i \quad\; j \end{array} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \\ \leftarrow j \\ \\ \\ \end{array}$$

$SL_n \subset GL_n \longrightarrow PGL_n$

with coroots

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & t & & & \\ & & & \ddots & & \\ & & & & t^{-1} & \\ & & & & & \ddots \; 1 \end{pmatrix} \qquad \alpha^\vee = \varepsilon_i - \varepsilon_j$$

$GL_n \qquad T = \mathbb{G}_m^n \qquad X \underset{=}{=} X(T) = \mathbb{Z}^n \qquad X^* = \mathbb{Z}^n \quad \left(\cong (\mathbb{Z}^n)^* \text{ via } \langle \varepsilon_i, e_j \rangle = \delta_{ij} \right)$

$SL_n \qquad T = \ker\left(\mathbb{G}_m^n \xrightarrow[\det]{} \mathbb{G}_m \right)$

$$\begin{array}{ccccc} & & \uparrow & & \uparrow \\ X & \leftarrow & X(GL_n) & \leftarrow & \mathbb{Z} \\ \| & & \| & & \\ \mathbb{Z}^n/\mathbb{Z}\cdot(1,\dots,1) & & \mathbb{Z}^n & & \end{array}$$

$X^* = \{ \beta \in \mathbb{Z}^n \mid \Sigma \beta_i = 0 \} \ni \varepsilon_i - \varepsilon_j$

$PGL_n \qquad T = \operatorname{coker}\left(\mathbb{G}_m \xrightarrow{\text{diag}} \mathbb{G}_m^n \right) \qquad X^* = \mathbb{Z}^n / \mathbb{Z}\cdot(1,\dots,1)$

$$\begin{array}{ccccc} \mathbb{Z} & \leftarrow & \mathbb{Z}^n & \leftarrow & X \\ \Sigma\lambda_i & \leftarrow\!\shortmid & \lambda & & \end{array}$$

$X = \{ \beta \in \mathbb{Z}^n \mid \Sigma \beta_i = 0 \}$

Langlands duals $\quad {}^L GL_n = GL_n, \quad {}^L SL_n = PGL_n$

Solvable radical = center

$Z(GL_n) = \mathbb{G}_m \qquad K^\times \cdot I$

semisimple $\begin{cases} Z(SL_n) \text{ finite } = n^{th} \text{ roots of unity} \\ Z(PGL_n) = 1 \end{cases}$

Describing the center: $Z(G) \subset T$, $= \ker \mathrm{Ad} : G \curvearrowright G$

$= \ker \mathrm{Ad} : G \curvearrowright \mathfrak{g}$

$= \ker \mathrm{Ad} : T \curvearrowright \mathfrak{g}$

$$0 \to Z(G) \to T \to \underset{= T(\mathrm{Ad}\, G)}{T'} \to 0$$

$G \to \underset{\text{image of } G \curvearrowright \mathfrak{g}}{\mathrm{Ad}\, G}$

$$0 \leftarrow X/Q \leftarrow X \leftarrow \underset{\text{root lattice}}{Q} \leftarrow 0$$

$$0 \leftarrow \underset{kX/Q}{\mathcal{O}(Z(G))} \leftarrow \underset{= kX}{\mathcal{O}(T)} \leftarrow \underset{= kQ}{\mathcal{O}(T')} \leftarrow 0$$

$\mathcal{O}(T)/I \to \mathcal{O}(Z(G))$

$\mathcal{O}(T)/I$ $\quad$ I generated by $x^\lambda - 1$ for $\lambda \in Q$

If X/Q is a lattice, e.g. for GL_n: $\quad X = \mathbb{Z}^n \quad Q = \{\beta \mid \sum \beta_i = 0\}$
$= \mathbb{Z} \cdot \{e_i - e_j\}$

$Z(G) = \mathbb{G}_m \qquad X/Q = \mathbb{Z}$ $\quad \beta \in X \mapsto \sum \beta_i$

In char 0, if X/Q is finite, $\underset{= Z(G)}{\mathrm{Spec}(kX/Q)}$ is the dual $\underset{\text{finite}}{}$ abelian group.

In char p, $Z(G)$ can be a <u>non-reduced group scheme</u>

$$1 \to Z(SL_n) \to SL_n \to PGL_n \to 0$$

$$\begin{array}{ccccccccc} 0 & \leftarrow & X/Q & \leftarrow & X & \leftarrow & Q = X(PGL_n) & \leftarrow & 0 \\ & & & & \| & & \| & & \\ 0 & \leftarrow & \mathbb{Z}/n\mathbb{Z} & \leftarrow & \mathbb{Z}^n/\mathbb{Z}\cdot(1,\dots,1) & & \mathbb{Z}\cdot\langle e_i - e_j\rangle & & \end{array}$$

Char 0 $\quad Z(SL_n) = \mu_n = \{n^{th} \text{ roots of unity}\}\cdot I$

Any char: $\quad k\cdot(\mathbb{Z}/n\mathbb{Z}) = k[t]/(t^n - 1)$ $\quad$ n-dimensional Hopf algebra, with $\underline{\Delta t = t\otimes t}$

Char p, $n=p$ $\quad k[t]/(t^p - 1) = k[t]/(t-1)^p$

Spec($\uparrow$) has only one point, with non-reduced scheme structure

$SL_p \to PGL_p$ is bijective, but not an isomorphism! $\quad SL_p$, PGL_p are both ordinary reduced alg. groups.

PGL_n has $X = Q$ $\quad Z$ trivial

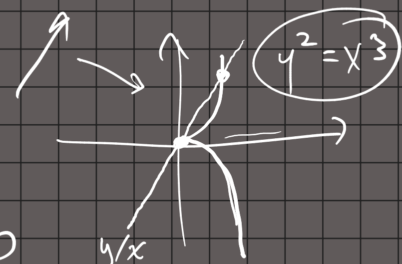
PGL_n is an adjoint group

(x,y)

$\updownarrow$

$t = y/x$

$(0,0) \mapsto 0$



$$k[t] \xleftarrow[x\mapsto t^2,\ y\mapsto t^3]{} k[x,y]/(y^2 - x^3)$$

X/Q finite : semisimple

dual to SL_n

Dual to adjoint group is the case $X^* = Q^\vee$ $\quad X/Q = (Q^\vee)^*/Q$

size = det of Cartan matrix

"Simply connected"

$\downarrow$

$Q \subset X$

$Q^\vee \subset X^*$

$(Q^\vee)^* \hookrightarrow Q$

$\cdots X \cdots$ $\quad Q/(Q^\vee)^*$

Cartan matrix $\langle \alpha_i^\vee, \alpha_j \rangle$ determines pairing of Q with $Q^\vee$

$$\begin{array}{ccc} Q^\vee & \hookrightarrow & Q^* \\ & \cap \; X^* \; \cup & \end{array}$$

Representations of SL_2

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad ad - bc = 1$

$\mathcal{O}(SL_2) = k[a,b,c,d]/(ad-bc-1)$ $\quad$ not graded, but filtered

F_n = pols of degree $\leq n$ $\qquad F_0 = k \cdot 1 \subset F_1 \subset \cdots$

$\dim F_n$ is the same as if we had $k[a,b,c,d]/(\deg 2)$

dim in each degree d

d	$\dim$
0	1
1	4
2	9
3	16
$\vdots$	
d	$(d+1)^2$

$\left\langle {4 \atop 2} \right\rangle = \binom{4+2-1}{2} = \binom{5}{2} = 10$

$- \left\langle {4 \atop 0} \right\rangle$ $\qquad 10-1$

$\left\langle {4 \atop 3} \right\rangle = \binom{6}{3} = 20$

$\left\langle {4 \atop 3} \right\rangle - \left\langle {4 \atop 1} \right\rangle = 20-4$

$$\binom{d+3}{3} - \binom{d+1}{3} = (d+1)^2$$

$SL_2 \curvearrowright V = \mathbb{C}^2$ and on $\mathcal{O}(\mathbb{C}^2) = k[x,y]$

$(x,y) \mapsto (x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad V_2 \quad a^2, 2ac, \ldots$

$x \mapsto ax + cy \qquad y \mapsto bx + dy$

sub into $f(x,y)$

$SL_2 \curvearrowright k[x,y]_d = S^d V^*$

$\quad\quad\quad\quad\quad\quad\quad\quad\;\; \| $

$\quad\quad\quad\quad\quad\quad\quad\quad\; V_d$ $\qquad \dim V_d = d+1 \longrightarrow (d+1)^2$ independent function on SL_2 :

$V_0 \quad 1$

$V_1 \quad a, b, c, d$

$x^2 \mapsto (ax+cy)^2 = a^2x^2 + 2ac\,xy + c^2y^2$

$xy \mapsto$

$y^2 \mapsto$

$= ?\, x^2 \quad ?\, xy \quad ?\, y^2$

matrix entries of V_d

$F_n \subset \mathcal{O}(SL_2)$ is $\bigoplus_{d \le n} \langle$ matrix entries of $V_d \rangle$

V_d is irreducible: $\quad U = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cong \mathbb{G}_a \subset SL_2$

Any invariant $0 \neq W \subseteq V_d$ contains a U-invariant vector:

$$f(x,y) \quad \begin{matrix} x \mapsto x \\ y \mapsto bx + y \end{matrix}$$

$$\downarrow$$

$$f(x, y+bx) = f(x,y) \Rightarrow f \text{ ind. of } y$$

$$\overset{\shortparallel}{f(x)}$$

x^d generates V_d, only U-invariant $\quad x^d$

$\Rightarrow V_d$ is irreducible.

$\Longrightarrow$ Sl_2 is reductive, and the V_d are all the finite-dimensional irr. alg. reps.

$$X = \mathbb{Z} \quad Q = 2\mathbb{Z}$$
$$X^* = \mathbb{Z} \quad Q^\vee = \mathbb{Z}$$

$T(SL_2) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cong \mathbb{G}_m \qquad$ roots $\pm\alpha \qquad \alpha = 2$ in $\mathbb{Z} = X(\mathbb{G}_m)$

$\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \mathfrak{g}_\alpha \qquad$ coroots $\pm\alpha^\vee$

$$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

$V_1 = V$

$V_2 = \mathfrak{g}$

(alternative in PGL_2)

irreps V_{2k}